Critical RSOS and Minimal Models I: Paths, Fermionic Algebras and Virasoro Modules

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Abstract

We consider $s\ell(2)$ minimal conformal field theories on a cylinder from a lattice perspective. To each allowed one-dimensional configuration path of the A_L Restricted Solid-on-Solid (RSOS) models we associate a physical state and a monomial in a finite fermionic algebra. The orthonormal states produced by the action of these monomials on the primary states $|h\rangle$ generate finite Virasoro modules with dimensions given by the finitized Virasoro characters $\chi_h^{(N)}(q)$. These finitized characters are the generating functions for the double row transfer matrix spectra of the critical RSOS models. We argue that a general energy-preserving bijection exists between the one-dimensional configuration paths and the eigenstates of these transfer matrices and exhibit this bijection for the critical and tricritical Ising models in the vacuum sector. Our results extend to $\mathbb{Z}_{L-1}$ parafermion models by duality.

1 Introduction

Over the last two decades there has been much progress in understanding the deep connections between conformal field theory (CFT) [?, ?] and integrable lattice models [?]. But there remain some intriguing mysteries. For example it is well-known, from the work of the Stony Brook group, that the Virasoro characters admit a fermionic quasi-particle interpretation [?] and this has led to many recent developments (see for example [?] and [?] and references therein). But what is the fermionic algebra underlying the structure of the finitized characters and what are the fermionic quasi-particles? There is also a well-known correspondence principle [?] which states that the Corner Transfer Matrix (CTM) one-dimensional configurational sums of the RSOS lattice models [?] in the off-critical Regimes III and II give rise to the finitized characters of the $s\ell(2)$ unitary minimal and $\mathbb{Z}_k$ parafermion models respectively. But off-criticality there is no conformal symmetry or Virasoro algebra. So how do these one-dimensional configurational sums appear in the treatment of the critical RSOS models where the conformal algebra really is present? On the other hand, workers in the field have been trying for a long time, to build [?] matrix representations of the Virasoro algebra based on paths and to make sense [?, ?] of a finitized Virasoro algebra. We address these questions in this series of papers building on work [?] on a fermionic interpretation of Baxter's Corner Transfer Matrix (CTM) paths.

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The layout of this paper is as follows. In Section 2, we define the critical RSOS models and their double-row transfer matrices. We also recall their relation to the minimal and parafermion CFTs. In Section 3, we introduce fermionic algebras and paths. We give the relation between physical states and fermionic paths and discuss fermionic states and finite Virasoro modules. The bijection between the one-dimensional RSOS configuration paths and eigenstates of the critical RSOS double row transfer matrices is exhibited for the critical and tricritical Ising models in Section 4. We conclude in Section 5 with a discussion of open questions for further research.

2 Critical RSOS Models and Conformal Field Theory

2.1 Critical RSOS models and double row transfer matrices

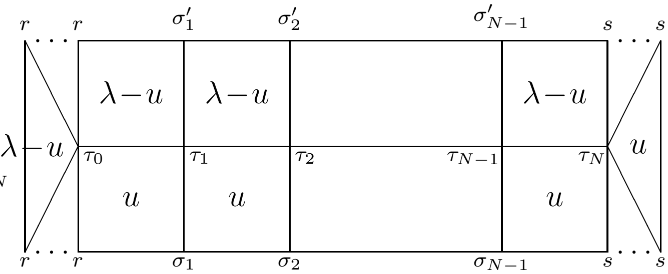
The critical A_L RSOS models [?] are exactly solvable models [?] on the square lattice. The Boltzmann weights associated to the elementary faces are

$$W\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, u\right) = \begin{matrix} {\scriptstyle d} & & {\scriptstyle c} \\ & \boxed{u} & \\ {\scriptstyle a} & & {\scriptstyle b} \end{matrix} = \sin(\lambda-u)\sin\lambda\,\delta_{a,c} + \sin u\sin\lambda\sqrt{S_aS_cS_bS_d}\,\delta_{b,d} \tag{2.1}$$

where the heights $a,b,c,d \in \{1,2,\ldots,L\}$ and u is the spectral parameter. The crossing parameter is $\lambda = (p-p')\pi p$ with p,p' coprime and $p' < p = L+1$. The crossing factors are $S_a = \sin a\lambda$ and the weights vanish if the heights on any edge do not differ by ± 1. For *unitary* models ($p' = p-1$), these Boltzmann weights are all nonnegative whereas, for *nonunitary* models ($p' < p-1$), some of these weights are negative.

Since the RSOS face weights satisfy the Yang-Baxter equation (YBE), these models are integrable for arbitrary complex u using commuting transfer matrix methods [?]. To work on a strip or cylinder, with specified boundary conditions (r_L, s_L) on the left and (r_R, s_R) on the right edges, it is necessary to introduce [?] double row transfer matrices. In this paper we will consider just the subset of boundary conditions with $(r_L, s_L) = (r,1)$ and $(r_R, s_R) = (1,s)$. We call these (r,s) type boundary conditions. With (r,s) boundary conditions, the double row transfer matrices are represented diagrammatically by

$$\boldsymbol{D}(u)_{\sigma,\sigma'} = \sum_{\tau_0,\ldots,\tau_N} \tag{2.2}$$



For integrability, the triangle boundary weights on the left and right must satisfy the boundary Yang-Baxter equation (BYBE). For each conformal boundary condition (r_L, s_L) or (r_R, s_R) on an edge, there is a corresponding integrable boundary condition [?] given by a set of triangle boundary weights satisfying the BYBE. For (r,s) type boundary conditions, the partition functions satisfy $Z_{(1,1)|(r,s)} = Z_{(r,1)|(1,s)}$. In the vacuum sector $(r_L, s_L) = (r_R, s_R) = (1,1)$, the triangle boundary weights vanish unless $\tau_0 = \tau_N = 2$ so the triangle boundary weights can simply be removed leaving the heights alternating between 1 and 2 on the left and right edges of the strip.

For boundary conditions of (r,s) type, the double row transfer matrices $\boldsymbol{D}(u)$ form a commuting family $[\boldsymbol{D}(u), \boldsymbol{D}(v)] = 0$. Consequently, they can be simultaneously diagonalized by the orthogonal matrix of eigenstates which are independent of u. For a suitable choice of parameters, the double row transfer matrices are real symmetric and positive definite. The actual form of the eigenstates changes under an orthogonal change of basis. Nevertheless, these eigenstates are characterised by their associated eigenvalues $D(u)$ which are *independent* of the choice of basis. These eigenvalues in turn can be studied analytically by Yang-Baxter techniques and are classified according to their patterns of zeros in the complex u-plane. Consequently, we can use the patterns of zeros to label the eigenstates.

The A_L RSOS models exhibit two distinct physical regimes. If $0 < u < \lambda$, the continuum scaling limit realizes the $s\ell(2)$ minimal models. Otherwise, if $\lambda - \pi/2 < u < 0$, the continuum scaling limit realizes the $\mathbb{Z}_{L-1}$ parafermions. The conformal data is obtained from the finite-size corrections to the eigenvalues of the double row transfer matrices. In making contact with CFT, the spectral parameter is usually specialized to its *isotropic* value, $u = \lambda/2$ for minimal models and $u = -\lambda$ for $\mathbb{Z}_{L-1}$ parafermions.

2.2 Minimal models and $\mathbb{Z}_k$ parafermions

The $s\ell(2)$ minimal models [?] $\mathcal{M}(p', p)$ with p, p' coprime have central charges

$$c = 1 - \frac{6\,(p - p')^2}{p\,p'} \tag{2.3}$$

The conformal weights are

$$h = h_{r,s} = (rp - sp')^2 - (p - p')^2 4pp', \qquad r = 1, 2, \ldots, p' - 1; \qquad s = 1, 2, \ldots, p - 1 \tag{2.4}$$

and the Virasoro characters are

$$\chi_h(q) = q^{-c/24+h}(q)_\infty \sum_{k=-\infty}^{\infty} \left(q^{k(kpp'+rp-sp')} - q^{(kp+s)(kp'+r)}\right) \tag{2.5}$$

where

$$(q)_n = \prod_{k=1}^{n}(1 - q^k) \tag{2.6}$$

The minimal models are unitary if $p - p' = \pm 1$. We consider only the diagonal A-type series with $p' < p$ and use the critical Ising $\mathcal{M}(3, 4)$, tricritical Ising $\mathcal{M}(4, 5)$ and Yang-Lee theories $\mathcal{M}(2, 5)$ as prototypical examples.

The $s\ell(2)$ $\mathbb{Z}_k$ parafermion models [?] have central charges

$$c = 2(k - 1)k + 2, \qquad k = 2, 3, \ldots \tag{2.7}$$

We consider only the diagonal A-type series and we use the $\mathbb{Z}_3$ or hard hexagon model [?, ?] as the prototypical example. The hard hexagon model is in the universality class of the 3-state Potts model so we refer to this as the 3-state Potts CFT. Generally, the characters of the $\mathbb{Z}_k$ models are string functions but, for the $\mathbb{Z}_3$ model, these are easily related to the Virasoro characters of the $\mathcal{M}(5, 6)$ model.

The minimal and $\mathbb{Z}_k$ parafermion models are rational and admit a finite number of primary fields $\phi(z) = \phi^{(h)}(z)$. These theories arise from the continuum scaling limit of the A_L RSOS models with $L = p - 1$ and $L = k + 1$ respectively.

2.3 Virasoro algebra and Virasoro states

The Virasoro algebra

$$\mathrm{Vir} = \langle L_n, n \in \mathbb{Z} \rangle \tag{2.8}$$

is an infinite dimensional complex Lie algebra associated with conformal symmetry. The generators L_n satisfy the commutation relations

$$[L_n, L_m] = (n-m)L_{n+m} + c12\, n(n^2-1)\delta_{n,-m} \tag{2.9}$$

where the central element c is the central charge. The Virasoro generators are the modes of the energy-momentum tensor

$$T(z) = \sum_{n\in\mathbb{Z}} L_n\, z^{-n-2} \tag{2.10}$$

On a cylinder with prescribed boundary conditions, which is the case of primary concern here, there is just one copy of the Virasoro algebra. For bulk theories on the torus, however, there is a second copy $\overline{\mathrm{Vir}}$ of Virasoro which is the antiholomorphic counterpart.

For rational CFTs, the Hilbert space $\mathcal{H}$ of states on which Vir acts is naturally decomposed into a finite direct sum of irreducible highest weight representations (Virasoro modules)

$$\mathcal{H} = \oplus_h \mathcal{V}_h \tag{2.11}$$

where the sum is over the conformal weights h of the primary fields $\phi(z) = \phi^{(h)}(z)$. The vacuum $|0\rangle$ and primary (highest weight) states $|h\rangle$ are characterised by

$$L_0|h\rangle = h|h\rangle; \qquad L_n|0\rangle = 0, \quad n \geq -1; \qquad L_n|h\rangle = 0, \quad n > 0 \tag{2.12}$$

Moreover, there is a one-to-one correspondence between primary fields and primary states induced by

$$\lim_{z\to 0} \phi^{(h)}(z)|0\rangle = |h\rangle \tag{2.13}$$

The vacuum state $|0\rangle$ with $h = 0$ corresponds to the identity operator.

The generically reducible highest weight representation of Vir (Verma module) is the linear span of Virasoro states in the canonical form

$$L_{-n_j}L_{-n_{j-1}}\ldots L_{-n_1}|h\rangle, \qquad n_j \geq n_{j-1} \geq \cdots \geq n_1 \geq 1 \tag{2.14}$$

If its maximal proper submodule is quotiented out, we are led to the irreducible Virasoro module $\mathcal{V}_h = \mathcal{V}_{c,h}$ and the states (2.14) are no longer linearly independent due to the existence of null vectors. The generic Virasoro module in the $h = 0$ vacuum sector is shown in Figure 1. Typically, for given c and h, some states at a given level enter in a vanishing non-trivial linear combination that is the null vector. Surprisingly, it seems that a complete set of linearly independent Virasoro states is not known even for the Ising model, although it is known [?] for the Yang-Lee theory $\mathcal{M}(2,5)$ and the whole family $\mathcal{M}(2, 2n+3)$, $n \geq 1$.

With reference to the vectors (2.14), the module $\mathcal{V}_h$ is graded according to the level

$$\mathcal{V}_h = \mathop{\oplus}_{\ell=0}^{\infty} \mathcal{V}_{h,\ell}, \qquad \ell = \sum_{i=1}^{j} n_i \tag{2.15}$$

The Virasoro character $\chi_h(q)$, which is the generating function for the spectrum of the Virasoro module $\mathcal{V}_h$, is

$$\chi_h(q) = \mathrm{Tr}_{\mathcal{V}_h} q^{L_0 - c/24} = q^{-c/24+h} \sum_{\ell=0}^{\infty} d_\ell \, q^\ell, \qquad d_\ell \geq 0 \tag{2.16}$$

where q is the modular parameter and the degeneracy $d_\ell = d_\ell^h = \dim \mathcal{V}_{h,\ell}$ is the dimension of the space of states at level ℓ.

1	$\vert 0\rangle$			
0	–			
q^2	$L_{-2}\vert 0\rangle$			
q^3	$L_{-3}\vert 0\rangle$			
$2q^4$	$L_{-4}\vert 0\rangle$	$L_{-2}^2\vert 0\rangle$		
$2q^5$	$L_{-5}\vert 0\rangle$	$L_{-3}L_{-2}\vert 0\rangle$		
$4q^6$	$L_{-6}\vert 0\rangle$	$L_{-4}L_{-2}\vert 0\rangle$	$L_{-3}^2\vert 0\rangle$	$L_{-2}^3\vert 0\rangle$
$4q^7$	$L_{-7}\vert 0\rangle$	$L_{-5}L_{-2}\vert 0\rangle$	$L_{-4}L_{-3}\vert 0\rangle$	$L_{-3}L_{-2}^2\vert 0\rangle$
$7q^8$	$L_{-8}\vert 0\rangle$	$L_{-6}L_{-2}\vert 0\rangle$	$L_{-5}L_{-3}\vert 0\rangle$	$L_{-4}^2\vert 0\rangle$
	$L_{-4}L_{-2}^2\vert 0\rangle$	$L_{-3}^2L_{-2}\vert 0\rangle$	$L_{-2}^4\vert 0\rangle$	
$8q^9$	$\dots$	$\dots$	$\dots$	$\dots$
$12q^{10}$	$\dots$	$\dots$	$\dots$	$\dots$

Figure 1: Virasoro module $\mathcal{V}_0$ of Virasoro states in the vacuum $h = 0$ sector. The generic Virasoro character is $\chi_0(q) = \prod_{n=2}^{\infty}(1-q^n)^{-1} = 1 + q^2 + q^3 + 2q^4 + 2q^5 + 4q^6 + 4q^7 + 7q^8 + 8q^9 + 12q^{10} + \cdots$. In this sector, there is a null vector $L_{-1}|0\rangle = 0$ at level $\ell = 1$. For the minimal theories $\mathcal{M}(p', p)$, further null vectors appear. For example, for the Ising model $\mathcal{M}(3, 4)$, there is one null vector at level 6 and 7 and two at level 8. For $\mathcal{M}(4, 5)$, the first null vector enters at level 12.

3 Fermionic Algebras and States

3.1 Fermionic algebras

Consider the N-step configuration paths $\{\sigma\}$ of the A_L RSOS models [?] as shown in Figures 2 and 3 with $\sigma_j \in \{1, 2, 3, \dots, L\}$ and $\sigma_{j+1} - \sigma_j = \pm 1$. In this context, applying conformal boundary conditions of (r, s) type means that $\sigma_0 = s$, $\sigma_N = r$ and $\sigma_{N+1} = r + 1$ where $s = 1, 2, \dots, L$ and $r = 1, 2, \dots, L - 1$. Alternatively, we can work with infinite paths which start at s and after N steps alternate between heights r and $r + 1$. Allowing for the $\mathbb{Z}_2$ height reversal symmetry, there are $12L(L - 1)$ distinct boundary conditions or sectors. In the case of the unitary minimal models $\mathcal{M}(L, L + 1)$, these are in one-to-one correspondence with the primary operators $\phi^{(h)}(z)$ with conformal weights in the Kac table

$$h = h_{r,s} = h_{L-r,L+1-s} = \big((L + 1)r - Ls\big)^2 - 14L(L + 1), \qquad 1 \leq r \leq L - 1, \quad 1 \leq s \leq L \tag{3.1}$$

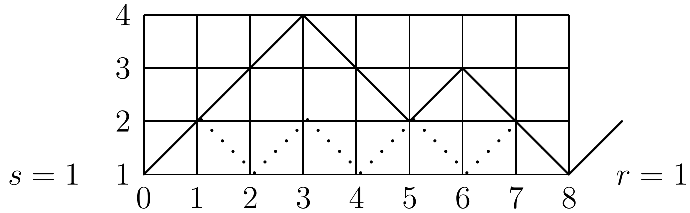


Figure 2: The $N = 8$ step RSOS configurational path $\sigma = \{1, 2, 3, 4, 3, 2, 3, 2, 1\}$ in the vacuum $(r, s) = (1, 1)$ sector of the tricritical Ising model $\mathcal{M}(4, 5)$ corresponding to the fermionic state $b_{-72}b_{-2}b_{-1}b_{-12}|0\rangle$. The energy of this path is $E(\sigma) = 12(1 + 2 + 4 + 7) = 7$. The groundstate vacuum path $|0\rangle$ is shown dotted.

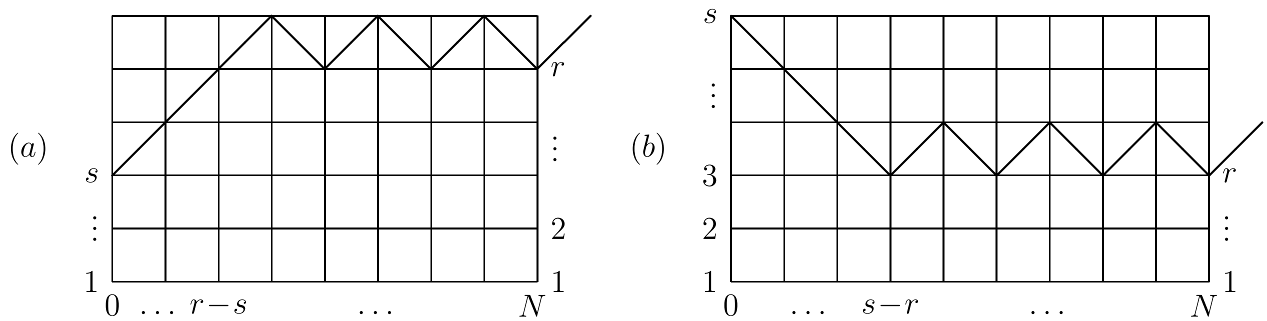


Figure 3: The N-step path σ_h associated with the primary state $|h\rangle$ of the (r, s) sector for (a) $s \le r$ and (b) $s > r$. The parity of N is fixed by $N = |r - s| \bmod 2$. Sectors of type (a) and (b) are related by the $\mathbb{Z}_2$ Kac table symmetry $(r, s) \equiv (L - r, L + 1 - s)$ under height reversal.

These RSOS paths were first introduced in the context of the Corner Transfer Matrices (CTMs) for the off-critical RSOS models but we argue in Section 4 that the same paths appear naturally at criticality in classifying the eigenstates of the critical RSOS double row transfer matrices. Each path is associated with a state $|\sigma\rangle$ with configurational energy

$$E(\sigma) = 12\sum_{j=1}^{N} j\, H(\sigma_{j-1}, \sigma_j, \sigma_{j+1}) \tag{3.2}$$

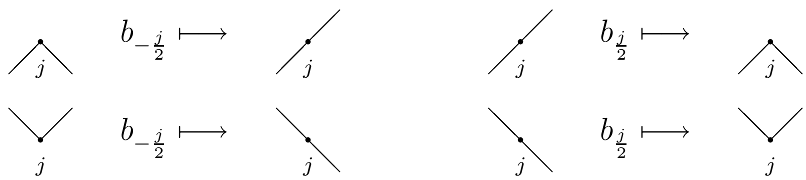
For the unitary minimal models $\mathcal{M}(L, L + 1)$ the elementary local excitation energy is

$$H(\sigma_{j-1}, \sigma_j, \sigma_{j+1}) = 12|\sigma_{j-1} - \sigma_{j+1}| = \begin{cases} 0, & \sigma_{j+1} = \sigma_{j-1} \\ 1, & \sigma_{j+1} - \sigma_{j-1} = \pm 2 \end{cases} \tag{3.3}$$

This description originates with the off-critical RSOS models in Regime III with q the elliptic modulus but here we apply it to the Regime III/IV critical RSOS models with q the modular parameter. The vacuum or groundstate $|0\rangle$ with zero energy is the path that alternates between heights 1 and 2. In the (r, s) sector, the primary or reference state $|h\rangle$ is the path σ_h with lowest energy as shown in Figure 3. The (r, s) sector is associated with the Virasoro module $\mathcal{V}_h$ with $h = h_{r,s}$. Specifically, the finitized Virasoro characters $\chi^{(N)}_{r,s}(q) = \chi^{(N)}_h(q)$ approach the corresponding Virasoro characters in the limit $N \to \infty$ with $N = |r - s| \bmod 2$

$$\chi^{(N)}_h(q) := q^{-c/24+h} \sum_{\{\sigma\}} q^{E(\sigma)-E(\sigma_h)} \to \chi_h(q), \qquad N \to \infty \tag{3.4}$$

Let us just consider the RSOS models related to unitary minimal models in the sector $h = h_{r,s}$. Then at each position j along an A_L path σ there is either a corner with energy 0 or a straight segment with energy 1 and these two possibilities are mutually exclusive. We regard an elementary excitation as the action of a fermionic operator $b_{\mp j/2} = b^h_{\mp j/2}$ that annihilates (creates) a corner at position j of σ and creates (annihilates) a straight segment at the same position of σ. The fermionic behaviour (Pauli exclusion principle) is suggested by the fact that at position j only one straight segment or corner can exist



$$b^2_{\pm\frac{j}{2}} = 0 \qquad (3.5)$$

The associated energy $H(\sigma_{j-1}, \sigma_j, \sigma_{j+1})$ is the eigenvalue of the fermion number operator

$$b_{-j2}b_{j2}|\sigma\rangle = H(\sigma_{j-1}, \sigma_j, \sigma_{j+1})|\sigma\rangle \qquad (3.6)$$

where in anticipation of unitarity we have set $b_{j/2} = b^\dagger_{-j/2}$.

The dual description originates with the RSOS models in the off-critical Regime II but here we apply it to the critical Regime I/II RSOS models describing $\mathbb{Z}_{L-1}$ parafermions. In the dual description the roles of the corners and straight segments are interchanged so that

$$H'(\sigma_{j-1}, \sigma_j, \sigma_{j+1}) = 1 - H(\sigma_{j-1}, \sigma_j, \sigma_{j+1}) = \begin{cases} 0, & \sigma_{j+1} - \sigma_{j-1} = \pm 2 \\ 1, & \sigma_{j+1} = \sigma_{j-1} \end{cases} \qquad (3.7)$$

Apart from a shift in the zero of energy, this involves an overall change of sign in the energy $E(\sigma)$. The groundstate $|0\rangle'$ with zero energy is now the saw-tooth path (see Figure 5) $\sigma = \{1, 2, \ldots, L-1, L, L-1, \ldots, 2, 1, 2, \ldots\}$. In this dual picture an elementary excitation annihilates a straight segment and creates a corner at position j of the path. The associated energy $H'(\sigma_{j-1}, \sigma_j, \sigma_{j+1})$ is the eigenvalue of the dual fermion number operator

$$b_{j2}b_{-j2}|\sigma\rangle = H'(\sigma_{j-1}, \sigma_j, \sigma_{j+1})|\sigma\rangle \qquad (3.8)$$

Comparison with (3.3) and (3.7) suggests that

$$b_{-j2}b_{j2} + b_{j2}b_{-j2} = 1 \qquad (3.9)$$

consistent with our fermionic interpretation. Guided by the Ising or free fermion $\mathcal{M}(3,4)$ case, we assume the additional fermionic relations $b_j b_k = -b_k b_j$, $j \neq -k$.

We now introduce a fermionic algebra generated by all the fermion operators

$$\mathcal{F} = \oplus_h \mathcal{F}_h, \qquad \mathcal{F}_h = \langle b^h_j,\ j \in \mathbb{Z}/2, j \neq 0\rangle \qquad (3.10)$$

There is an independent copy $\mathcal{F}_h$ of the fermion algebra in each sector h with

$$b^h_j\, b^{h'}_k = b^h_j\, b^h_k\, \delta_{h,h'} \qquad (3.11)$$

We usually work in a fixed sector and suppress the index h. The generators in each sector satisfy the usual Canonical Anticommutator Relations (CAR) for fermions as well as the involutive property for real fermions

$$\{b_j, b_k\} = \delta_{j,-k}, \qquad b_{-j} = b^\dagger_j = b^T_j \qquad (3.12)$$